

ENTIRE s -HARMONIC FUNCTIONS ARE AFFINE

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ABSTRACT. In this paper, we prove that solutions to the equation $(-\Delta)^s u = 0$ in \mathbb{R}^N , for $s \in (0, 1)$, are affine. This allows us to prove the uniqueness of the Riesz potential $|x|^{2s-N}$ in Lebesgue spaces.

1. INTRODUCTION

The classical Liouville theorem for harmonic functions states that *a bounded harmonic function in \mathbb{R}^N is constant*, see for instance the particularly short proof by E. Nelson in [12]. The stronger version of it states that a nonnegative harmonic function on \mathbb{R}^N is constant. In the case of the fractional Laplacian $(-\Delta)^s$, for $s \in (0, 1)$, (see Section 2), the strong form of the Liouville theorem holds as well and was proved by K. Bogdan et al. in [4]. Applications of Liouville theorems for nonlocal operators in the study of nonlocal elliptic systems of equations can be found in [1, 8, 13, 14]. The aim of this paper is to classify all s -harmonic functions in \mathbb{R}^N , thereby obtaining the Liouville theorem for the fractional Laplacian as a particular case.

Theorem 1.1. *Every s -harmonic function in \mathbb{R}^N is affine, and constant if $s \in (0, 1/2]$.*

The proof of this theorem is mainly based on a Cauchy-type estimate for the derivatives of an s -harmonic function. More precisely, given $s \in (0, 1)$, $\gamma \in \mathbb{N}^N$ and a function u which is s -harmonic in the ball $B(0, R)$, we have the estimate

$$|D^\gamma u(0)| \leq CR^{2s-|\gamma|} \int_{|y| \geq R/4} |u(y)| |y|^{-N-2s} dy, \quad (1.1)$$

for some positive constant C depending only on N, γ and s , see Section 3. This estimate is obtained from the Poisson kernel representation formula for s -harmonic functions. We refer to Section 2 for more details.

In the following, we denote by $\mathcal{D}'(\mathbb{R}^N)$ the dual of $C_c^\infty(\mathbb{R}^N)$ endowed with the usual topology. An iteration argument based on Theorem 1.1 allows to state the following result.

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Theorem 1.2. *Assume that $s \in (0, 1)$ and let u be a solution to the equation*

$$(-\Delta)^s u = P \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

where P is a polynomial. Then u is affine and $P = 0$.

Another consequence of the main theorem which is of independent interest is the following result.

Corollary 1.3. *Let $p \in [1, \infty)$ and $u \in L^p(\mathbb{R}^N)$ be such that*

$$(-\Delta)^s u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Then $u \equiv 0$.

Combining Corollary 1.3 and the Hardy-Littlewood-Sobolev inequality, we have a uniqueness result.

Corollary 1.4 (Uniqueness of Riesz potential). *Let $s \in (0, 1)$, $1 < p < \frac{N}{2s}$ and $f \in L^p(\mathbb{R}^N)$. Then there exists a unique $u \in L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$ such that*

$$(-\Delta)^s u = f \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

and u is given by

$$u(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}} dy,$$

where

$$\alpha_{N,s} = \pi^{N/2} 2^{2s} \frac{\Gamma(s)}{\Gamma((N-2s)/2)}.$$

The paper is organized as follows. In Section 2 we collect some basic facts concerning the fractional Laplacian $(-\Delta)^s$ and s -harmonic functions. Finally, in Section 3 we prove the Cauchy-type estimate (1.1), the main result and its corollaries.

Note added in proof: We mention that after this paper was submitted, Liouville-type results for a class of nonlocal operators were proved in [7] and [9] using Fourier transform.

2. PRELIMINARIES

This section is devoted to recall some basic notions about s -harmonic functions. We refer the reader to [6, Section 3]. Let \mathcal{L}_s^1 denote the space of all measurable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}} dx < \infty.$$

For functions $\varphi \in C^2(\mathbb{R}^N) \cap \mathcal{L}_s^1$, the fractional Laplacian $(-\Delta)^s$ is defined by

$$-(\Delta)^s \varphi(x) = C_{N,s} \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{\varphi(y) - \varphi(x)}{|y-x|^{N+2s}} dy \quad \text{for all } x \in \mathbb{R}^N, \quad (2.1)$$

where $C_{N,s} = s(1-s)\pi^{-N/2}4^s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(2-s)}$.

For $u \in \mathcal{L}_s^1$, the expression $(-\Delta)^s u$ defines a distribution on every open set $\Omega \subset \mathbb{R}^N$ by

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^N} u(x) (-\Delta)^s \varphi(x) dx \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

In the case where $(-\Delta)^s u = 0$ in $\mathcal{D}'(\Omega)$, we will say that u is s -harmonic in Ω .

We note that affine functions u belong to \mathcal{L}_s^1 if $s > 1/2$ and constant functions u belong to \mathcal{L}_s^1 if $s \in (0, 1/2]$. Moreover, by using (2.1), in both cases, we can see that $(-\Delta)^s u(x) = 0$ for every $x \in \mathbb{R}^N$. Furthermore, thanks to [6, Lemma 3.3], we have $(-\Delta)^s u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$.

The fractional Laplacian has an explicit Poisson kernel with respect to the ball $B(x, r)$ (see [3]). It is given by

$$P_r(x, y) = \begin{cases} \beta_{N,s} \frac{(r^2 - |x|^2)^s}{(|y|^2 - r^2)^s} |y - x|^{-N} & \text{for } |x| < r, |y| > r, \\ 0 & \text{otherwise,} \end{cases} \quad (2.2)$$

where $\beta_{N,s} = \Gamma(N/2)\pi^{-N/2-1} \sin(s\pi)$. Therefore (see also [4]), if u is s -harmonic in Ω then for every ball $B(a, r) \subset\subset \Omega$ we have

$$u(x) = \int_{\mathbb{R}^N} P_r(x - a, y - a) u(y) dy \quad \text{for all } x \in B(a, r).$$

We now consider the regularization of P_r as in [6]. To this end, we pick a function $\phi \in C_c^\infty(1, 4)$ such that $\int_{\mathbb{R}} \phi(r) dr = 1$ and define $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\Psi(y) = \int_1^4 P_r(0, y) \phi(r) dr = \beta_{N,s} |y|^{-N} \int_{\min(1, |y|)}^{\min(4, |y|)} r^{2s} (|y|^2 - r^2)^{-s} \phi(r) dr.$$

Observe that, if $|y| \leq 1$ then $(0, |y|) \cap (1, 4) = \emptyset$ and thus

$$\Psi(y) = 0 \quad \text{for every } y \in B(0, 1). \quad (2.3)$$

Furthermore, as shown e.g. in [6, Lemma 3.11], we have $\Psi \in C^\infty(\mathbb{R}^N)$. Moreover, for every $\gamma \in \mathbb{N}^N$ there holds

$$|D^\gamma \Psi(y)| \leq C |y|^{-N-2s-|\gamma|} \quad \text{for every } y \in \mathbb{R}^N \setminus \{0\}, \quad (2.4)$$

where $C = C(N, \gamma, s)$ denotes, here and in the following, a positive constant depending only on N, γ and s .

We define $\Psi_{r_0}(y) = r_0^{-N} \Psi(y/r_0)$, for $y \in \mathbb{R}^N$ and $r_0 > 0$. Then, for any s -harmonic function u in an open set $\Omega \subset \mathbb{R}^N$, we have

$$u(x) = u \star \Psi_{r_0}(x) \quad \text{for all almost every } x \in \Omega_{4r_0}, \quad (2.5)$$

where $\Omega_{4r_0} = \{x \in \Omega : \text{dist}(x, \mathbb{R}^N \setminus \Omega) > 4r_0\}$, see [10, Lemma 2.6] or [6, Page 65]. We will therefore assume, in the sequel, that s -harmonic functions in some open set are smooth in that set.

3. PROOF OF THE MAIN RESULT AND ITS CONSEQUENCES

The following result (from which we will derive our main result) can be seen as a nonlocal version of the Cauchy estimate for bounded harmonic functions, see e.g. [2, Chapter 2].

Lemma 3.1. *For every $\gamma \in \mathbb{N}^N$, there exists a constant $C > 0$ only depending on N , γ and s such that for every function u which is s -harmonic in $B(0, R)$,*

$$|D^\gamma u(0)| \leq C R^{2s-|\gamma|} \int_{|y| \geq R/4} |u(y)| |y|^{-N-2s} dy.$$

Proof. Let u be an s -harmonic function in $B(0, R)$ and $r_0 \in (0, R/4)$. Then by (2.5) we have

$$u(x) = u \star \Psi_{r_0}(x) \quad \text{for all } x \in B(0, R - 4r_0).$$

By (2.3), (2.4) and the dominated convergence theorem, we deduce that

$$D^\gamma u(0) = u \star D^\gamma \Psi_{r_0}(0) \quad \text{for all } \gamma \in \mathbb{N}^N.$$

Using once more (2.3) and (2.4), we get

$$|D^\gamma u(0)| = \left| \int_{|y| \geq r_0} u(y) D^\gamma \Psi_{r_0}(-y) dy \right| \leq C r_0^{2s} \int_{|y| \geq r_0} |u(y)| |y|^{-N-2s-|\gamma|} dy.$$

It follows that

$$|D^\gamma u(0)| \leq C r_0^{2s-|\gamma|} \int_{|y| \geq r_0} |u(y)| |y|^{-N-2s} dy.$$

Letting $r_0 \rightarrow R/4$, we get the desired estimate. \square

As a consequence of Lemma 3.1, we have the following result.

Corollary 3.2. *Let Ω be a nonempty open set of \mathbb{R}^N such that $\Omega \neq \mathbb{R}^N$. Then for every $\gamma \in \mathbb{N}^N$, there exists a constant $C > 0$ only depending on N , γ and s such that for every function u which is s -harmonic in Ω ,*

$$|D^\gamma u(x)| \leq C \delta_\Omega^{2s-|\gamma|}(x) \int_{|y| \geq \frac{\delta_\Omega(x)}{4}} |u(y)| |y|^{-N-2s} dy \quad \text{for all } x \in \Omega,$$

where $\delta_\Omega(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega)$.

Proof. By assumption, $\delta_\Omega(x) < \infty$ for every $x \in \Omega$. Since $B(x, \delta_\Omega(x)) \subset \Omega$, applying Lemma 3.1 to the function $y \mapsto u(y + x)$ and $R = \delta_\Omega(x)$, we get the desired result. \square

Proof of Theorem 1.1.

Let $x \in \mathbb{R}^N$ and $r > 0$. We apply Corollary 3.2 with $\Omega = B(x, r)$ and $|\gamma| \geq 2s$. Then, letting $r \rightarrow \infty$, we get $|D^\gamma u(x)| = 0$ for every $|\gamma| \geq 2s$ and $x \in \mathbb{R}^N$. The proof of the theorem is thus completed. \square

Remark 3.3. *It is well known that there are smooth functions u — hence in $L^1_{loc}(\mathbb{R}^N)$ — satisfying $\Delta u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ for $N \geq 2$ which are not polynomials. Therefore a natural question arises: does there exist a larger space of distributions, strictly containing \mathcal{L}_s^1 , where the fractional laplacian is appropriately defined and where there are nontrivial entire s -harmonic functions which are not affine?*

Proof of Theorem 1.2. The proof will be done by induction. Suppose ℓ is the degree of P . Assume that $\ell = 0$ so that P is a constant. Let $h \in \mathbb{R}^N$ and $u_h(x) = u(x+h) - u(x)$. It is clear that $u_h \in \mathcal{L}_s^1$. In addition $(-\Delta)^s u_h = 0$. It follows from Theorem 1.1 that $\partial_{i,j} u_h(0) = 0$ and therefore $\partial_{i,j} u(h) = \partial_{i,j} u(0)$ for every $h \in \mathbb{R}^N$. This implies that u is a second order polynomial and since it belongs to \mathcal{L}_s^1 , it is affine.

Now assume that the result holds true for a polynomial of degree up to $\ell \geq 0$ and suppose that $(-\Delta)^s u = P_{\ell+1}$, a polynomial of degree $\ell+1$. Then, for $h \in \mathbb{R}^N$, using the binomial formula we can see that $(-\Delta)^s u_h = P_{\ell,h}$, where $P_{\ell,h}$ is a polynomial of degree ℓ . It follows from our assumption that u_h is affine for any $h \in \mathbb{R}^N$. This again implies that u is a second order polynomial and thus an affine function, since it belongs to \mathcal{L}_s^1 . \square

Proof of Corollary 1.3. We just note that $L^q(\mathbb{R}^N) \subset \mathcal{L}_s^1$ for every $q \in [1, \infty]$ by Hölder's inequality. \square

Proof of Corollary 1.4. We define the function $\tilde{u}(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}} dy$. Let $f_n \in C_c^\infty(\mathbb{R}^N)$ be such that $f_n \rightarrow f$ in $L^p(\mathbb{R}^N)$. Define $u_n(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f_n(y)}{|x-y|^{N-2s}} dy$. By the Hardy-Littlewood-Sobolev inequality (see [11, Theorem 4.3]), we have $u_n \rightarrow \tilde{u}$ in $L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$. In particular, $u_n \rightarrow \tilde{u}$ in \mathcal{L}_s^1 by Hölder's inequality. Thanks to [5, Lemma 5.3], we have $(-\Delta)^s u_n = f_n$ in $\mathcal{D}'(\mathbb{R}^N)$. Passing to the limit as $n \rightarrow \infty$, we deduce that $(-\Delta)^s \tilde{u} = f$ in $\mathcal{D}'(\mathbb{R}^N)$. Finally, if $u \in L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$ is an arbitrary solution to $(-\Delta)^s u = f$ in $\mathcal{D}'(\mathbb{R}^N)$ then $(-\Delta)^s(u - \tilde{u}) = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. We thus conclude, from Corollary 1.3, that $u = \tilde{u}$. \square

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